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Expo. Math. 26 (2008) 187–196

EXPOSITIONES MATHEMATICAE

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On some properties of the gamma function

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Received 4 May 2007; received in revised form 28 June 2007

Abstract

In this paper we prove a complete monotonicity theorem and establish some upper and lower bounds for the gamma function in terms of digamma and polygamma functions.

MSC 2000: primary 33B15; secondary 26D07

Keywords: Gamma function; Digamma function; Polygamma functions; Complete monotonicity; Riemann zeta-function

1. Introduction

The gamma function Γ, defined for **Re** $z > 0$ by the improper integral

$$\Gamma(z) = \int_0^\infty u^{z-1} \mathrm{e}^{-u} \, \mathrm{d}u,$$

was introduced into analysis in the year 1729 by Leonard Euler [15] while seeking a generalization of the factorial $n!$ for non-integral values of n, and subjected to intense study by many eminent mathematicians of the nineteenth and early twentieth centuries and continues to interest the present generation. The logarithmic derivative ψ of the gamma function is known as the psi or digamma function, that is, it is given by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

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doi:10.1016/j.exmath.2007.10.001

for positive real numbers x. The derivatives $\psi', \psi'', \psi''', \ldots$ of the digamma function are called polygamma functions in the literature. The following series and integral representations are well known for the psi and polygamma functions:

$$\psi(u+1) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+u}\right) (u > -1), \tag{1.1}$$

where $\gamma = 05772156649\ldots$ is the Euler–Mascheroni constant, and

$$(-1)^{n-1}\psi^{(n)}(u) = n!\sum_{k=0}^{\infty}\frac{1}{(u+k)^{n+1}} = \int_0^{\infty}\frac{t^n \mathrm{e}^{-ut}}{1-\mathrm{e}^{-t}}\,\mathrm{d}t, \tag{1.2}$$

for $n = 1, 2, 3, \ldots$. See these and further properties of these functions Chapter 1 of [8] and [1, p. 260].

The gamma function is one of the most important special functions and has many applications in many fields of science, for example, analytic number theory, statistics and physics. See the very useful paper of Srinivasan [25] for the historical background and basic properties of the gamma function. In particular, in recent years many authors have studied this function and they obtained many remarkable inequalities; see [2–7,9,10,14,16,19,20,22–24] and the references therein. It is the aim of this paper to continue the study of this important function and to provide several new inequalities for it.

Our first theorem provides a complete monotonicity property of the gamma function. Recall that a function f is completely monotonic in an interval I if f has derivatives of all orders in I which alternate in sign, that is $(-1)^n f^{(n)}(x) \geqslant 0$ for all $x \in I$ and $n=0, 1, 2, 3, \ldots$. If this inequality is strict for all $x \in I$ and all non-negative integers n, then f is said to be strictly completely monotonic. Completely monotonic functions have many applications in different fields of science, for example, they have applications in probability theory [12,17,21], potential theory [11], physics [13] and numerical analysis [18]. An important result characterizing completely monotonic functions is the Hausdorff–Bernstein–Widder theorem. This theorem states that f is completely monotonic if and only if

$$f(x) = \int_0^{\infty} \mathrm{e}^{-xt}\,\mathrm{d}\mu(t),$$

where μ is a non-negative measure on $[0, \infty)$ such that this integral converges for all $x > 0$, see [26, Theorem 12b, p. 161].

2. Main results

Theorem 2.1. *Set for $x > 0$*

$$F_a(x) = \log(\Gamma(x)) - x\log x + x - \frac{1}{2}\log(2\pi) + \frac{1}{2}\psi(x) + \frac{1}{6(x-a)}. \tag{2.1}$$

Then $F_a(x)$ is completely monotonic on $(0, \infty)$ if and only if $a \geqslant 1/4$ and $-F_b(x)$ is completely monotonic if and only if $b \leqslant 0$.

Proof. Let $x > 0$ and $a \geqslant 0$. Differentiation gives

$$F_a'(x) = \psi(x) - \log x + \frac{1}{2}\psi'(x) - \frac{1}{6(x-a)^2},$$

and

$$F_a''(x) = \psi'(x) - \frac{1}{x} + \frac{1}{2}\psi''(x) + \frac{1}{3(x-a)^3}.$$

Using (1.2) and the integral representations

$$\frac{1}{x} = \int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}t, \qquad \frac{1}{(x-a)^3} = \frac{1}{2}\int_0^\infty t^2\mathrm{e}^{(a-x)t}\,\mathrm{d}t,$$

we obtain

$$F_a''(x) = \frac{1}{6}\int_0^\infty \mathrm{e}^{-xt}\frac{P_a(t)}{1-\mathrm{e}^{-t}}\,\mathrm{d}t, \tag{2.2}$$

where

$$P_a(t) = -6 + 6t - 3t^2 + 6\mathrm{e}^{-t} + t^2\mathrm{e}^{at} - t^2\mathrm{e}^{(a-1)t}. \tag{2.3}$$

The following asymptotic formulas are well known, see [1, p. 257, 259, 260]:

$$\log\Gamma(x) \sim \left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \frac{1}{12x} + \cdots, \tag{2.4}$$

$$\psi(x) \sim \log x - \frac{1}{2x} - \frac{1}{12x^2} + \cdots . \tag{2.5}$$

From these formulas we get

$$\lim_{x\to\infty} F_a(x) = \lim_{x\to\infty} F_a'(x) = 0. \tag{2.6}$$

Let $a \geqslant 1/4$. Since the mapping $a \to P_a(t)$ is strictly increasing we have

$$\begin{aligned} P_a(t) \geqslant P_{1/4}(t) &= \mathrm{e}^{-t}(6 - 6\mathrm{e}^t + 6t\mathrm{e}^t - 3t^2\mathrm{e}^t + t^2\mathrm{e}^{5t/4} - t^2\mathrm{e}^{t/4}) \\ &= 16\mathrm{e}^{-t}\sum_{n=5}^\infty \left(\frac{t}{4}\right)^n \frac{\omega_n}{n!}, \end{aligned}$$

where

$$\omega_n = (n-1)[6.4^{n-2} - 3.n.4^{n-2} + n.5^{n-2} - n].$$

A direct computation gives $\omega_5 > 0$ and $\omega_6 > 0$. Since $(5/4)^{n-2} > 3$ and $6.4^{n-2} > n$ for $n \geqslant 7$, this reveals that $\omega_n > 0$ for $n \geqslant 5$. Hence, $P_a(t) > 0$, which yields by Eq. (2.2) that $F_a''(x) > 0$. Thus, $F_a''(x)$ is strictly completely monotonic for $a \geqslant 1/4$. Also, from Eq. (2.6) we get $F_a(x) > 0$ and $F_a'(x) < 0$, which yields that $F_a(x)$ is completely monotonic. Now we assume that $F_a(x)$ is strictly completely monotonic. This reveals that $F_a(x)>0$ for $x>0$ and so we obtain from Eq. (2.1) that

$$a \geqslant \lim_{x\to\infty}\left[x+\frac{1}{6\log\Gamma(x)-6x\log x+6x-3\log(2\pi)+3\psi(x)}\right].$$

Using the asymptotic formulas (2.4) and (2.5) this limit becomes, after simplifying,

$$\begin{aligned}a &\geqslant \lim_{x\to\infty}\left[x+\frac{1}{6\log\Gamma(x)-6x\log x+6x-3\log(2\pi)+3\psi(x)}\right]\\ &= \lim_{x\to\infty}\frac{x}{4x+1}=1/4.\end{aligned}$$

This proves the first part of our first theorem. From Eq. (2.3) we get

$$e^t P_0(t)=(-6+6t-2t^2)e^t+6-t^2=Q(t), \text{ say.}$$

A short calculation gives $Q(0)=Q'(0)=Q''(0)=0$ and

$$Q'''(t)=-(6t+2t^2)e^t<0,$$

which proves that $Q(t)<0$ for $t>0$, and hence $P_0(t)<0$. So, we conclude from Eq. (2.2) that $-F_0''(x)$ is strictly completely monotonic. From Eq. (2.6) we obtain $-F_0'(x)<0$ and $-F_0(x)>0$. Thus, $-F_0(x)$ is strictly completely monotonic. If $-F_b(x)$ (with $b\geqslant 0$) is strictly completely monotonic on $(0,\infty)$, then we have $-F_b(x)>0$ and hence

$$b\leqslant \lim_{x\to 0}\frac{1}{6\log\Gamma(x)-6x\log x+6x-3\log(2\pi)+3\psi(x)}=0. \tag{2.7}$$

It should be remarked that for $0<a<1/4$, $P_a(t)$ behaves like $(a-1/4)t^4$ for small t and like t^2e^{at} for large t and so has a change of sign. Hence by the "only if" part of Hausdorff–Bernstein–Widder representation theorem, neither $F_a(t)$ nor its derivative can be completely monotonic for these values of a. □

This completes the proof of Theorem 2.1.

Corollary 2.2. *Let $x>0$. Then we have*

$$\exp\left(-\frac{1}{2}\psi(x)-\frac{1}{6(x-\alpha)}\right)<\frac{\Gamma(x)}{x^x e^{-x}\sqrt{2\pi}}<\exp\left(-\frac{1}{2}\psi(x)-\frac{1}{6(x-\beta)}\right), \tag{2.8}$$

with best possible constants $\alpha=1/4$ and $\beta=0$.

Proof. From Theorem 2.1 we have for $x>0$

$$F_0(x)<0<F_{1/4}(x),$$

which is equivalent to Eq. (2.8) with $\alpha=1/4$ and $\beta=0$. If the left-hand side of Eq. (2.8) holds, as we show in the proof of Theorem 2.1, this leads to $\alpha\geqslant 1/4$. If we assume that there exists a positive number β such that the right-hand side of Eq. (2.8) holds for all $x>0$, then we obtain

$$\beta \leqslant \lim_{x\to 0} \frac{1}{6\log\Gamma(x) - 6x\log x + 6x - 3\log(2\pi) + 3\psi(x)} = 0.$$

Hence, the best possible constants in Eq. (2.8) are given by $\alpha = 1/4$ and $\beta = 0$. □

Remark 2.3. In [2] it was proved that for $x>0$ the double inequality

$$\exp\left(-\frac{1}{2}\psi(x+\alpha)\right) < \frac{\Gamma(x)}{x^x e^{-x}\sqrt{2\pi}} < \exp\left(-\frac{1}{2}\psi(x+\beta)\right) \tag{2.9}$$

holds with best possible constants $\alpha = 1/3$ and $\beta = 0$. It is clear that the right-hand side of Eq. (2.8) improves the right-hand side of Eq. (2.9).

The following theorem gives new bounds for the gamma function in terms of ψ'-function

Theorem 2.4. *For all positive real numbers x the following double inequality holds*:

$$\exp\left(-\gamma x + \frac{x^2}{2}\psi'(\alpha(x))\right) < \Gamma(x+1) < \exp\left(-\gamma x + \frac{x^2}{2}\psi'(\beta(x))\right),$$

where $\alpha(x) = 1 + x/3$ *and* $\beta(x) = x/\sqrt{2x - 2\log(x+1)}$.

Proof. Integrating both sides of Eq. (1.1) from $u=0$ to $u=x$, we obtain

$$\log\Gamma(x+1) = -\gamma x + \sum_{n=1}^{\infty}\left(\frac{x}{n} - \log(n+x) + \log n\right). \tag{2.10}$$

By Taylor's theorem there exists a $\theta = \theta(n)$ depending on x such that $0 < \theta(n) < x$ and

$$\log(n+x) - \log n = \frac{x}{n} - \frac{x^2}{2(n+\theta(n))^2}. \tag{2.11}$$

Substituting this into (2.10) we deduce

$$\log\Gamma(x+1) = -\gamma x + \frac{x^2}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1+\theta(n+1))^2}. \tag{2.12}$$

From Eq. (2.11) we obtain that

$$\theta(n) = \frac{x}{\sqrt{2x/n - 2\log(1+x/n)}} - n. \tag{2.13}$$

Now we shall show that $\theta(n)$ is strictly increasing for all $n \geqslant 1$ and $x>0$. After differentiation of both sides of Eq. (2.13) with respect to n, if we put $n = x/t$ we find that

$$\theta'(x/t) = \frac{t^3/(1+t) - (2t - 2\log(t+1))^{3/2}}{(2t-2\log(t+1))^{3/2}}. \tag{2.14}$$

Hence, in order to prove that θ is strictly increasing in n, it suffices to see

$$G(t) = \frac{t^6}{(t+1)^2} - 8(t - \log(t+1))^3 > 0.$$

Differentiation gives

$$G'(t) = \frac{2t}{(t+1)^3}\varphi(t),$$

where

$$\varphi(t) = 2t^5 + 3t^4 - 12(t+1)^2(t - \log(t+1))^2.$$

An easy computation gives $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$ and

$$\varphi'''(t) = \frac{\sigma(t)}{t+1},$$

where

$$\sigma(t) = 120t^3 - 96t^2 - 96t + (144t + 96)\log(1+t).$$

We can easily see that $\sigma(0) = \sigma'(0) = \sigma''(0) = 0$ and

$$\sigma'''(t) = \frac{720t^3 + 2160t^2 + 2016t + 480}{(t+1)^3} > 0.$$

From the facts $\sigma(0) = \sigma'(0) = \sigma''(0) = 0$ this implies that $\sigma(t) > 0$ for all $t > 0$. Hence, $\varphi'''(t) > 0$ for all $t > 0$. Using the relation $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$, we conclude $\varphi(t) > 0$ and so $G'(t) > 0$. Therefore by (2.14) we have $\theta'(x/t) > 0$. In conclusion, $\theta(n)$ is strictly increasing for all $n \geqslant 1$. Thus, by virtue of Eq. (2.12) we obtain

$$-\gamma x + \frac{x^2}{2}\psi'(1 + \theta(\infty)) < \log \Gamma(x+1) < -\gamma x + \frac{x^2}{2}\psi'(1 + \theta(1)). \tag{2.15}$$

Using l'Hospital's rule we can easily see that

$$\theta(\infty) = \lim_{n\to\infty} \theta(n) = x/3.$$

From Eq. (2.13) we have

$$\theta(1) = \frac{x}{\sqrt{2x - 2\log(x+1)}} - 1.$$

Replacing these in Eq. (2.15) and then simplifying the resulting identity, we complete the proof of Theorem 2.4. □

Theorem 2.5. *For all $x > 0$ we have*

$$\exp\left(-\gamma x + \frac{x^2}{2}\zeta(2) + \frac{x^3}{6}\psi''(\alpha(x))\right) < \Gamma(x+1)$$
$$< \exp\left(-\gamma x + \frac{x^2}{2}\zeta(2) + \frac{x^3}{6}\psi''(\beta(x))\right),$$

where $\alpha(x)=x[3(\log(x+1)-x+x^2/2)]^{-1/3}$, *and* $\beta(x)=1+x/4$. *Here* ζ *is the Riemann zeta function defined for* **Re** $s>1$ *by*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Proof. Applying Taylor's Theorem to $g(t)=\log t$ up to the third derivative on the interval $[k,k+x]$ we get

$$\log(k+x)-\log k=\frac{x}{k}-\frac{x^2}{2k^2}+\frac{x^3}{3(k+\omega(k))^3}, \tag{2.16}$$

where $\omega=\omega(k)$ is a real number between 0 and x. Using this relation into the series representation (2.10) we deduce

$$\begin{aligned}\log\Gamma(x+1)&=-\gamma x+\sum_{k=1}^{\infty}\left(\frac{x^2}{2k^2}-\frac{x^3}{3(k+\omega(k))^3}\right)\\&=-\gamma x+\frac{x^2\zeta(2)}{2}-\frac{x^3}{3}\sum_{k=0}^{\infty}\frac{1}{(k+1+\omega(k+1))^3}.\end{aligned} \tag{2.17}$$

From Eq. (2.16) we get

$$\omega(k)=\left[\frac{3}{x^3}\log(1+x/k)-\frac{3}{kx^2}+\frac{3}{2k^2x}\right]^{-1/3}-k.$$

Now we shall prove that ω is strictly increasing. By differentiating we get

$$\omega'(k)=\frac{x^4}{k^4}\frac{1}{(1+x/k)}\left[3\log(1+x/k)-\frac{3x}{k}+\frac{3x^2}{2k^2}\right]^{-4/3}-1.$$

If we set $t=x/k$ this becomes

$$\omega'(x/t)=\frac{t^4}{1+t}\left[3\log(1+t)-3t+\frac{3t^2}{2}\right]^{-4/3}-1.$$

So, to prove that $\omega'(k)>0$ it is sufficient to show that

$$H(t)=\left(\log(1+t)-t+\frac{t^2}{2}\right)^4-\frac{t^{12}}{81(1+t)^3}<0.$$

Differentiation of H gives

$$H'(t)=\frac{t^2}{1+t}\sigma_1(t),$$

where

$$\sigma_1(t) = 4\left(\log(1+t) - t + \frac{t^2}{2}\right)^3 - \frac{3t^{10}+4t^9}{27(1+t)^3}.$$

Differentiation of σ_1 gives

$$\sigma_1'(t) = \frac{t^2}{1+t}\sigma_2(t),$$

where

$$\sigma_2(t) = 12\left(\log(1+t) - t + \frac{t^2}{2}\right)^2 - \frac{7t^8+18t^7+12t^6}{9(1+t)^3}.$$

If we differentiate σ_2 we obtain that

$$\sigma_2'(t) = \frac{t^2}{1+t}\sigma_3(t),$$

where

$$\sigma_3(t) = 24\left(\log(1+t) - t + \frac{t^2}{2}\right) - \frac{35t^6+128t^5+162t^4+72t^3}{9(1+t)^3}.$$

If we differentiate once more we get

$$\sigma_3'(t) = -\frac{105t^6+250t^5+154t^4}{9(1+t)^4} < 0.$$

This implies that $\sigma_3(t)$ is strictly decreasing for all $t > 0$. Hence, we have $\sigma_3(t) < \sigma_3(0) = 0$, $\sigma_2(t)$ is strictly decreasing, $\sigma_2(t) < \sigma_2(0) = 0$. So, $\sigma_1(t)$ is strictly decreasing, implying $\sigma_1(t) < \sigma_1(0) = 0$. Thus we conclude that H is decreasing. This gives $H(t) < H(0) = 0$ for all $t > 0$. This proves that ω is strictly increasing for all $t > 0$. Thus by the help of Eq. (2.17) we find that

$$\begin{aligned} &-\gamma x + \frac{x^2\zeta(2)}{2} - \frac{x^3}{3}\sum_{k=0}^{\infty}\frac{1}{3(k+1+\omega(1))^3} < \Gamma(x+1) \\ &\quad < -\gamma x + \frac{x^2\zeta(2)}{2} - \frac{x^3}{3}\sum_{k=0}^{\infty}\frac{1}{3(k+1+\omega(\infty))^3}. \end{aligned} \tag{2.18}$$

It is clear that

$$\omega(1) = \left[\frac{3}{x^3}\log(1+x) - \frac{3}{x^2} + \frac{3}{2x}\right]^{-1/3} - 1.$$

If we apply l'Hospital rule successively we get

$$\lim_{k\to\infty}\omega(k) = x/4.$$

This proves Theorem 2.5 by the help of Eq. (2.18). $\square$

Following the same method used in the proof of Theorem 2.5 we can prove the following Theorem:

Theorem 2.6. *For all $x>0$ the following inequalities hold*:

$$\exp\left(-\gamma x+\frac{x^2}{2}\zeta(2)-\frac{x^3}{3}\zeta(3)+\frac{x^4}{24}\psi'''(a(x))\right)<\Gamma(x+1)$$

$$<\exp\left(-\gamma x+\frac{x^2}{2}\zeta(2)-\frac{x^3}{3}\zeta(3)+\frac{x^4}{24}\psi'''(b(x))\right),$$

where $a(x)=1+x/5$ *and* $b(x)=x(4x-2x^2+\frac{4x^3}{3}-4\log(x+1))^{-1/4}$.

In the light of Theorems 2.4–2.6 and some other observations we conjecture the following:

Conjecture. *For all* $x>0$ *and* $m=2,3,4,\ldots$

$$(-1)^m\gamma x+(-1)^m\sum_{k=2}^{m}\frac{(-1)^{k-1}\zeta(k)}{k}x^k+\frac{(-1)^{m-1}x^{m+1}}{(m+1)!}\psi^{(m)}(\alpha(x))$$

$$<(-1)^{m-1}\log\Gamma(x+1)$$

$$<(-1)^m\gamma x+(-1)^m\sum_{k=2}^{m}\frac{(-1)^{k-1}\zeta(k)}{k}x^k+\frac{(-1)^{m-1}x^{m+1}}{(m+1)!}\psi^{(m)}(\beta(x))$$

holds, *where*

$$\alpha(x)=1+\frac{x}{m+2}$$

and

$$\beta(x)=x\left((-1)^m(m+1)\left(\log(1+x)-\sum_{k=1}^{m}\frac{(-1)^{k-1}x^k}{k}\right)\right)^{-1/(m+1)}.$$

Acknowledgements

The author would like to thank to the referee for his/her valuable corrections and comments which improved the original version of this manuscript.

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